(b) Energy - Time uncertainty relation

- $\Delta t \, \Delta E \simeq \hbar$ is not about time uncertainty.

  ∵ Remember, time is just a parameter!

- def. Correlation amplitude

$$C(t) \equiv \langle \alpha | \alpha ; t \rangle \qquad \| \; t_0 = 0$$

: Resemblance between the state kets at different times.

For $|\alpha\rangle = \sum_n c_n |n\rangle$ $\quad \| \; |n\rangle$: energy eigenkets.

$$C(t) = \langle \alpha | \mathcal{U}(t) | \alpha \rangle$$

$$= \sum_n c_n^* \langle n | \cdot \mathcal{U}(t) \cdot \sum_{n'} c_{n'} |n'\rangle$$

$$= \sum_n |c_n|^2 \exp\left[-\frac{i E_n t}{\hbar} \right]$$

at $t=0$, $C(t)$; as t increases, $C(t)$ decreases if E_n is random.

(?)

(There are a lot deeper theories..)

- If we consider a large system with a quasi-continuous spectrum,

$$\sum_n \to \int dE \, \underbrace{\rho(E)}_{\text{density of states}} , \qquad c_n \to g(E_n)$$

$$\Rightarrow \quad C(t) = \int dE \, |g(E)|^2 \rho(E) \exp\left[\frac{-iEt}{\hbar} \right]$$

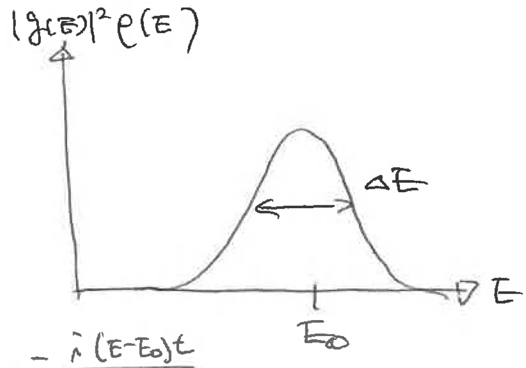
$\|$ normalization condition $\int dE \, |g(E)|^2 \rho(E) = 1$.

• If Energy is well defined,

$$E_0 = \langle H \rangle = \sum_n |c_n|^2 E_n \quad : \text{ time-indep.}$$

$$\rightarrow \int dE\, |g(E)|^2 \rho(E)\, E = E_0$$

meaning that $|g(E)|^2 \rho(E)$ is peaked at $E = E_0$!



• Coming back to $C(t)$,

$$C(t) = \int dE\, |g(E)|^2 \rho(E)\, e^{-\frac{iEt}{\hbar}}$$

$$= e^{-\frac{iE_0 t}{\hbar}} \int dE\, |g(E)|^2 \rho(E)\, e^{-\frac{i(E-E_0)t}{\hbar}}$$

• $e^{-\frac{i(E-E_0)t}{\hbar}} \rightarrow$ when (short time) $\frac{(E-E_0)t}{\hbar} \ll 1$, $\int dE \cdots e^{-i(E-E_0)t/\hbar} \rightarrow$ "finite"

$\quad\rightarrow$ when $t \gg \frac{\hbar}{\Delta E}$ $\quad \| \quad |E-E_0| \lesssim \Delta E$

$$\int dE \cdots \underbrace{e^{-i(E-E_0)t/\hbar}}_{\text{random phase !}} \rightarrow 0 .$$

$\Rightarrow$ "characteristic" time

$$t \simeq \frac{\hbar}{\Delta E} .$$

(above which it loses the initial state !

$\Rightarrow$ time - energy uncertainty relation

$$\boxed{\Delta t\, \Delta E \simeq \hbar}$$

(It has nothing to do with incompatible observables.)

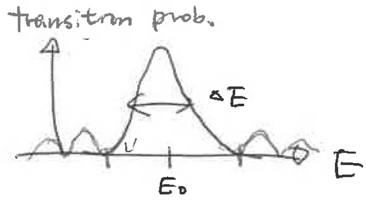
※ Δt : the time scale to retain the information of the original state.

ΔE : the relevant energy spread in the system.

* Another interpretation of $\Delta t \Delta E \sim \hbar$ in the perturbation theory.

Δt : duration of "drive" (ex. measurement time in spectroscopy.)

ΔE : spectral width of the transition obtained in exp. ($\approx$ uncertainty). i.e.



Now, this is not given but measured.

2.2 Schrödinger vs. Heisenberg picture

(1) Two interpretations of the unitary transformation.

Consider

$$\langle\beta|X|\alpha\rangle \xrightarrow{U} \langle\beta|U^{\dagger}XU|\alpha\rangle$$

- interpretation 1.

$|\alpha\rangle \longrightarrow U|\alpha\rangle$: the state is changed.

$X \longrightarrow X$: the operator is unchanged.

- interpretation 2.

$|\alpha\rangle \longrightarrow |\alpha\rangle$: the state is unchanged.

$X \longrightarrow U^{\dagger}XU$: the operator is changed.

In fact, the interpretation 2 is more classical-mechanics friendly!

In the classical mechanics,

$$X \to X+\delta X, \quad L \to L+\delta L, \quad \cdots$$

$\Longrightarrow$ [QM ver.2] $\hat{X} \to \hat{X}+\delta X$, $\hat{L} \to \hat{L}+\delta L$, $\cdots$

ex. $\mathcal{J}(\delta x)$: infinitesimal position translation.

$$\tilde{x} \to \left(1+\frac{i\tilde{p}\,\delta x}{\hbar}\right)\tilde{x}\left(1-\frac{i\tilde{p}\,\delta x}{\hbar}\right)$$

$$= \tilde{x} + \frac{i}{\hbar}[\hat{p}\,\delta x, \tilde{x}]$$

$$= \tilde{x} + \delta x$$

$\Rightarrow$ measurement

$$\langle \tilde{x} \rangle = \langle \tilde{x} \rangle + \langle \delta x \rangle$$

$\mathcal{J}(\delta x)|x\rangle$

"the same" result!

$\mathcal{J}^{\dagger}\tilde{x}\mathcal{J}$

Interpretation 1 $\to$ "Schrödinger picture"

The state ket is evolving.

Interpretation 2 $\to$ "Heisenberg picture"

The operator is evolving.

(2) State kets and Observables in the two pictures

$$A^{(H)}(t) = \mathcal{U}^{\dagger}(t)\,A^{(S)}\,\mathcal{U}(t) \qquad \Bigg\| \qquad A^{(H)}(0) = A^{(S)}$$

Heisenberg

Schrödinger.

$$|\alpha, t_0=0; t\rangle_H = |\alpha, t_0=0\rangle$$

$$|\alpha, t_0=0; t\rangle_S = \mathcal{U}(t)|\alpha, t_0=0\rangle .$$

$\langle A \rangle$: unchanged.

(3) Heisenberg Equation of Motion

- We need an equation for the time-evolution of an "operator".

$$\frac{dA^{(H)}}{dt} = \frac{d}{dt}\left(U^\dagger A^{(s)} U\right) = \frac{\partial U^\dagger}{\partial t} A^{(s)} U + U^\dagger A^{(s)} \frac{\partial U}{\partial t}$$

$$= -\frac{1}{i\hbar} U^\dagger H \underbrace{\hat{}}_{U U^\dagger} A^{(s)} U + U^\dagger A^{(s)} \underbrace{\hat{}}_{U U^\dagger} \cdot \frac{1}{i\hbar} H U$$

$$= \frac{1}{i\hbar}\left[- U^\dagger H U \underbrace{A^{(H)}}_{\equiv A(t)} + A^{(H)} U^\dagger H U\right]$$

$$= \frac{1}{i\hbar}\left[A^{(H)}, \underbrace{U^\dagger H U}_{= H}\right]$$

$= H$ ($\because [U, H] = 0$.)

$$\Rightarrow \boxed{\frac{dA^{(H)}}{dt} = \frac{1}{i\hbar}\left[A^{(H)}, H\right]} + \left(\frac{\partial A^H}{\partial t}\right)$$

Heisenberg EOM

$\hookrightarrow$ when A^H has an explicit time-dependence.

* Notation note.

Often, we write $\left[\begin{array}{l} A^{(H)} \equiv A(t) \\ A^{(s)} = A \end{array}\right]$.

Classical - Quantum correspondence

$$\frac{dA}{dt} = [A, H]_{\text{classical}}$$

$\uparrow$ Poisson Bracket.

$$\frac{[\ ,\ ]^{QM}}{i\hbar} \longleftrightarrow [\ ,\ ]_{\text{classical}}$$

(4) Free particles ; Ehrenfest's Theorem.

$$H = \frac{\vec{\tilde{p}}^{\,2}}{2m} = \frac{1}{2m}\left(\tilde{p}_x^{\,2} + \tilde{p}_y^{\,2} + \tilde{p}_z^{\,2}\right).$$

Heisenberg EOM

‖ NOTE: All operators are in the Heisenberg picture !.

① $$\frac{d\tilde{p}_i}{dt} = \frac{1}{i\hbar}[\tilde{p}_i, H] = 0 \quad : \text{ conserved !.}$$

② $$\frac{d\tilde{x}_i}{dt} = \frac{1}{i\hbar}[\tilde{x}_i, H] = \frac{1}{i\hbar}\frac{1}{2m}\, i\hbar \frac{\partial}{\partial \tilde{p}_i}\left(\sum_{j=1}^{3} \tilde{p}_j^{\,2}\right).$$

$$= \frac{\tilde{p}_i}{m} = \frac{\tilde{p}_i(0)}{m}. \quad (\text{invariant}).$$

‖ $[\tilde{x}_i, F(\vec{p})] = i\hbar \frac{\partial F}{\partial \tilde{p}_i}$

$[\tilde{p}_i, G(\vec{\tilde{x}})] = -i\hbar \frac{\partial G}{\partial \tilde{x}_i}$

$$\Rightarrow \tilde{x}_i(t) = \tilde{x}_i(0) + \frac{\tilde{p}_i(0)}{m}\cdot t.$$

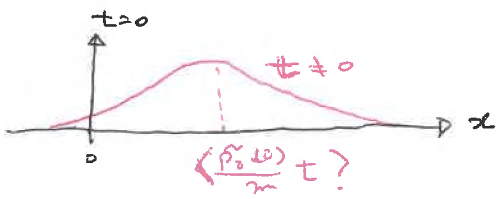
It looks like a classical dynamics, BUT

note that ...

$$[\tilde{x}_i(0), \tilde{x}_j(0)] = 0.$$

$$[\tilde{x}_i(t), \tilde{x}_j(0)] = \left[\frac{\tilde{p}_i(0)}{m}t, x_j(0)\right] = -\frac{i\hbar t}{m} \neq 0$$

∴ $\hat{x}$ operator spreads over distance in time.



uncertainty relation

$$\langle(\Delta\tilde{x}_i)^2\rangle_t \langle(\Delta\tilde{x}_i)^2\rangle_0 \geq \frac{\hbar^2 t^2}{4m^2}.$$

Now, adding a potential $V(\vec{x})$,

$$H = \frac{\tilde{\mathbb{P}}^2}{2m} + V(\vec{x})$$

EOM :
$$\frac{d\tilde{p}_i}{dt} = \frac{1}{i\hbar}\left[\tilde{p}_i, V(\vec{x})\right] = -\frac{\partial}{\partial x_i} V(\vec{x})$$

$$\frac{d\tilde{x}_i}{dt} = \frac{\tilde{p}_i}{m}.$$

also,
$$\frac{d^2\tilde{x}_i}{dt^2} = \frac{1}{i\hbar}\left[\frac{\tilde{p}_i}{m}, H\right] = \frac{1}{m}\frac{d\tilde{p}_i}{dt}.$$

$$\therefore \quad m\frac{d^2\vec{x}}{dt} = -\nabla V(\vec{x}) \quad \therefore$$

QM ver. Newton's second law!

for expectation values, // $|\alpha\rangle$ is t-indep. in the Heisenberg picture!

$$m\frac{d^2}{dt^2}\langle\vec{x}\rangle = \frac{d\langle\vec{p}\rangle}{dt} = -\langle\nabla V(\vec{x})\rangle.$$

NOTE THAT it's not $V(\langle x\rangle)$.

"Ehrenfest theorem".

(The center of a wave packet moves like a classical particle.)

Valid only in the Heisenberg picture.

~~independent~~ of the pictures.

(5) Base Kets and Transition amplitudes

	Schrödinger	Heisenberg
State ket	Moving.	Stationary
Observable	Stationary	Moving.
Base ket	Stationary	Moving oppositely.

- Base kets in the Schrödinger picture

Operator : time-independent

$\Rightarrow \quad A|a\rangle = a|a\rangle$

- In the Heisenberg picture

$A^{(H)}(t) = U^{\dagger} A U$

Thus. $\underset{U^{\dagger}}{\wedge} A \underset{UU^{\dagger}}{\wedge} |a\rangle = a \underset{U^{\dagger}}{\wedge} |a\rangle$ becomes

$\Rightarrow \quad A^{(H)}(t)\left(U^{\dagger}|a\rangle\right) = a\left(U^{\dagger}|a\rangle\right).$

$\Rightarrow \quad \underline{|a,t\rangle_H \equiv U^{\dagger}|a\rangle.}$: Base kets in the Heisenberg picture.

time-dependent.

- Time-evolution of $|a,t\rangle_H$.

$i\hbar\frac{\partial}{\partial t}|a,t\rangle_H = i\hbar\frac{\partial}{\partial t}U^{\dagger}|a\rangle = -HU^{\dagger}|a\rangle.$

$\Rightarrow \quad \underline{i\hbar\frac{\partial}{\partial t}|a,t\rangle_H = -H|a,t\rangle_H}$

moving in an opposite way!

- expansion coefficient $C_a(t)$

  - Schrödinger picture :

    $|\alpha,t\rangle = \sum_a C_a(t)|a\rangle. \Rightarrow C_a(t) = \overset{\text{base bra}}{\langle a|} \cdot \underbrace{U|\alpha,t=0\rangle}_{\text{state ket}}.$

    $\|$

  - Heisenberg picture :

    $|\alpha\rangle = \sum_a C_a(t)|a,t\rangle_H \Rightarrow C_a(t) = \underbrace{\langle a|U}_{\text{base bra}} \cdot \underbrace{|\alpha\rangle}_{\text{state ket}}.$

$\Rightarrow$ Transition probability

* The temporal Heisenberg inequality

- Ehrenfest theorem:

$$\frac{d}{dt}\langle A\rangle_\varphi = \frac{1}{i\hbar}\langle [A,H]\rangle_\varphi \qquad \| \ \langle \cdot \rangle_\varphi = \langle \varphi(t)|\cdot|\varphi(t)\rangle$$

- uncertainty relation : $\langle(\Delta A)^2\rangle_\varphi \langle(\Delta B)^2\rangle_\varphi \geq \frac{1}{4}|\langle[A,B]\rangle_\varphi|^2$

$\langle(\Delta A)^2\rangle_\varphi \equiv (\Delta_\varphi A)^2.$

Let's put H into B!

$$\rightarrow \quad \Delta_\varphi H \, \Delta_\varphi A \geq \frac{1}{2}|\langle[A,H]\rangle_\varphi| = \frac{1}{2}\hbar \left|\frac{d}{dt}\langle A\rangle_\varphi\right|$$

If we define the time $\tau_\varphi(A)$ as

$$\frac{1}{\tau_\varphi(A)} \equiv \left|\frac{d\langle A\rangle_\varphi}{dt}\right| \frac{1}{\Delta_\varphi A},$$

then τ_φ = characteristic time for the expectation value of A to change by $\Delta_\varphi A$.

$$\Rightarrow \quad \Delta_\varphi H \, \tau_\varphi(A) \geq \frac{1}{2}\hbar \quad \Rightarrow \quad \Delta E \Delta t \gtrsim \frac{1}{2}\hbar$$

Energy spread — characteristic evolution time.

2.3 Simple Harmonic oscillator

(1) Energy eigenkets. (Dirac's operator method)

$$H = \frac{\tilde{p}^2}{2m} + \frac{1}{2}m\omega^2\tilde{x}^2 = \hbar\omega(\tilde{a}^+\tilde{a} + \tfrac{1}{2})$$

$$\equiv \hbar\omega(\tilde{N} + \tfrac{1}{2}).$$

def. annihilation operator

$$\tilde{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\tilde{x} + i\frac{\tilde{p}}{m\omega}\right)$$

creation operator

$$\tilde{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\tilde{x} - i\frac{\tilde{p}}{m\omega}\right).$$

$$\Rightarrow \quad \tilde{x} = \frac{x_0}{\sqrt{2}}(\tilde{a} + \tilde{a}^+), \qquad \tilde{p} = i\frac{\hbar}{\sqrt{2}x_0}(-\tilde{a} + \tilde{a}^+)$$

$$\tilde{N} = \tilde{a}^+\tilde{a} \qquad \| \ x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

→ Commutation relation $[\tilde{a}, \tilde{a}^+] = 1$